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N- Fractional Calculus and $n(\in Z^{+})th$ Derivatives of Some Logarithmic Functions

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Abstract

In this article N- fractional calculus and *n-th* derivatives of logarithmic functions

$$\log((z-b)^m - c), \quad (m \in \mathbf{Z}_0^+) \quad \text{and} \quad \log(z^2 + 2az + d)$$

are reported.

That is, we have the following, for example.

(i) $$(\log((z-b)^m - c))_\gamma = -e^{-i\pi\gamma} m(z-b)^{-\gamma}\Gamma(\gamma) \times \sum_{k=0}^{\infty} \frac{\Gamma(mk+\gamma)}{\Gamma(\gamma)\Gamma(mk+1)}\left(\frac{c}{(z-b)^m}\right)^k$$

$$(|\Gamma(\gamma)|,\ |\Gamma(mk+\gamma)| < \infty),$$

and

(ii) $$(\log((z-b)^m - c))_n = (-1)^{n+1} m(z-b)^{-n}\Gamma(n) \times \sum_{k=0}^{\infty} \frac{\Gamma(mk+n)}{\Gamma(n)\Gamma(mk+1)}\left(\frac{c}{(z-b)^m}\right)^k$$

$$(n \in \mathbf{Z}^+),$$

where

$$(z-b)^m - c \neq 0, 1, \quad \textit{and} \quad |c/(z-b)^m| < 1 .$$

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D (z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\,d\zeta \qquad (\nu \notin \mathbf{Z}^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in \mathbf{Z}^+), \tag{2}$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for C_- , $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+ ,

$\zeta \ne z$, $z \in \mathbf{C}$, $\nu \in \mathbf{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(I I) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^ν be*

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}}\right) \qquad (\nu \notin \mathbf{Z}^-), \quad [\,\textit{Refer to}\ (1)\,] \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \qquad (m \in \mathbf{Z}^+), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \qquad (\alpha, \beta \in \mathbf{R}), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in \mathbf{R}\} \tag{6}$$

is an Abelian product group (having continuous index ν *) which has the inverse transform operator* $(N^\nu)^{-1} = N^{-\nu}$ *to the fractional calculus operator* N^ν *, for the* *function* f *such that* $f\in F=\{f;\ 0\neq|f_\nu|<\infty, \nu\in \boldsymbol{R}\}$, *where* $f=f(z)$ *and* $z\in \boldsymbol{C}$. (*vis.* $-\infty<\nu<\infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α.)

Theorem B. " F.O.G. $\{N^\nu\}$ " *is an " Action product group which has continuous index* ν *" for the set of* F . (*F.O.G.* ; Fractional calculus operator group) [3]

Theorem C. *Let*

$$S:=\{\pm N^\nu\}\cup\{0\}=\{N^\nu\}\cup\{-N^\nu\}\cup\{0\} \quad (\nu\in \boldsymbol{R}). \tag{7}$$

Then the set S *is a commutative ring for the function* $f\in F$ *, when the identity*

$$N^\alpha+N^\beta=N^\gamma \quad (N^\alpha, N^\beta, N^\gamma\in S) \tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(i) $$((z-c)^b)_\alpha=e^{-i\pi\alpha}\frac{\Gamma(\alpha-b)}{\Gamma(-b)}(z-c)^{b-\alpha} \quad \left(\left|\frac{\Gamma(\alpha-b)}{\Gamma(-b)}\right|<\infty\right),$$

(ii) $$(\log(z-c))_\alpha=-e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)|<\infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha}=-e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c) \quad (|\Gamma(\alpha)|<\infty),$$

where $z-c\neq 0$ for (i) and $z-c\neq 0,1$ for (ii), (iii) ,

§ 1. Preliminary

Theorem D. below for the fractional calculus of a logarithmic function is reported by K. Nishimoto (cf. J. Frac. Calc. Vol. 29, May (2006), p. 40.).

Theorem D. *We have*

(i) $$(\log((z-b)^\beta-c))_\gamma=-e^{-i\pi\gamma}\beta(z-b)^{-\gamma}\Gamma(\gamma)$$
$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\gamma)\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k \quad \left(|\Gamma(\gamma)|, \left|\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}\right|<\infty\right), \tag{1}$$

and

(ii) $$(\log((z-b)^\beta-c))_m=(-1)^{m+1}\beta(z-b)^{-m}\Gamma(m)$$
$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+m)}{\Gamma(m)\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k \quad (m\in \boldsymbol{Z}^+), \tag{2}$$

where

$$(z-b)^\beta-c\neq 0,1, \quad \textit{and} \quad |c/(z-b)^\beta|<1.$$

§ 2. N- Fractional Calculus of Functions

$$\log((z-b)^m - c)$$

Theorem 1. *We have*

(i) $$(\log((z-b)^m - c))_\gamma = -e^{-i\pi\gamma} m(z-b)^{-\gamma}\Gamma(\gamma)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(mk+\gamma)}{\Gamma(\gamma)\Gamma(mk+1)}\left(\frac{c}{(z-b)^m}\right)^k \quad (1)$$

$$(|\Gamma(\gamma)|,\ |\Gamma(mk+\gamma)| < \infty),$$

and

(ii) $$(\log((z-b)^m - c))_n = (-1)^{n+1} m(z-b)^{-n}\Gamma(n)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(mk+n)}{\Gamma(n)\Gamma(mk+1)}\left(\frac{c}{(z-b)^m}\right)^k \quad (n \in \mathbf{Z}^+), \quad (2)$$

where

$$m \in \mathbf{Z}_0^+, \quad (z-b)^m - c \neq 0, 1, \quad \textit{and} \quad |c/(z-b)^m| < 1.$$

Proof of (i). Set $\beta = m$ in Theorem D. (i) in Preliminary we have (1) clearly, under the conditions stated before.

Proof of (ii). Set $\gamma = n$ in (1), we have then (2).

Corollary 1. *We have*

(i) $$(\log((z-b)^2 - c))_\gamma = -e^{-i\pi\gamma} 2(z-b)^{-\gamma}\Gamma(\gamma)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(2k+\gamma)}{\Gamma(\gamma)\Gamma(2k+1)}\left(\frac{c}{(z-b)^2}\right)^k \quad (3)$$

$$(|\Gamma(\gamma)|,\ |\Gamma(2k+\gamma)| < \infty),$$

and

(ii) $$(\log((z-b)^2 - c))_n = (-1)^{n+1} 2(z-b)^{-n}\Gamma(n)$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(2k+n)}{\Gamma(n)\Gamma(2k+1)}\left(\frac{c}{(z-b)^2}\right)^k \quad (n \in \mathbf{Z}^+), \quad (4)$$

where

$$(z-b)^2 - c \neq 0, 1, \quad \textit{and} \quad |c/(z-b)^2| < 1.$$

Proof. Set $m = 2$ in Theorem 1.

Theorem 2. *We have the following identity.*

(i) $$2\sum_{k=0}^{\infty}\frac{\Gamma(2mk+\gamma)}{\Gamma(\gamma)\Gamma(2mk+1)}T^{k}=\sum_{k=0}^{\infty}\frac{\Gamma(mk+\gamma)}{\Gamma(\gamma)\Gamma(mk+1)}\{(T^{1/2})^{k}+(-T^{1/2})^{k}\} \quad (5)$$

$$(\ |\Gamma(mk+\gamma)|<\infty\ ),$$

and

(iI) $$2\sum_{k=0}^{\infty}\frac{[2mk+1]_{n-1}}{\Gamma(n)}T^{k}=\sum_{k=0}^{\infty}\frac{[mk+1]_{n-1}}{\Gamma(n)}\{(T^{1/2})^{k}+(-T^{1/2})^{k}\} \quad (6)$$

$$(\ n\in Z^{+}\ ),$$

where

$$T=\frac{c}{(z-b)^{2m}},\qquad |T|<1,\qquad m\in Z^{+}\ , \quad (7)$$

$$[\lambda]_{k}=\lambda(\lambda+1)\cdots(\lambda+k-1)=\Gamma(\lambda+k)/\Gamma(\lambda)\ \text{ with }\ [\lambda]_{0}=1.$$

(Notation of Pochhammer).

Proof of (i). We have

$$\log(\,(z-b)^{2m}-c)=\log((z-b)^{m}-\sqrt{c})+\log((z-b)^{m}+\sqrt{c})\,. \quad (8)$$

$$(\ (z-b)^{m}\pm\sqrt{c}\neq 0,\ 1\ ).$$

Operate N^{γ} to the both sides of (8), we have then

$$(\log((z-b)^{2m}-c))_{\gamma}=(\log((z-b)^{m}-\sqrt{c}))_{\gamma}+(\log((z-b)^{m}+\sqrt{c}))_{\gamma}\,. \quad (9)$$

Now we have

$$(\log((z-b)^{2m}-c))_{\gamma}=-e^{-i\pi\gamma}2m(z-b)^{-\gamma}\Gamma(\gamma)\sum_{k=0}^{\infty}\frac{\Gamma(2mk+\gamma)}{\Gamma(\gamma)\Gamma(2mk+1)}T^{k} \quad (10)$$

$$(\ |\Gamma(2mk+\gamma)|<\infty\ ),$$

and

$$(\log((z-b)^{m}-\sqrt{c}))_{\gamma}=-e^{-i\pi\gamma}m(z-b)^{-\gamma}\Gamma(\gamma)\sum_{k=0}^{\infty}\frac{\Gamma(mk+\gamma)}{\Gamma(\gamma)\Gamma(mk+1)}(T^{1/2})^{k} \quad (11)$$

$$(\ |\Gamma(mk+\gamma)|<\infty\ ),$$

by Theorem D. (i), respectively.

Then we obtain (i) from (9), applying (10) and (11), under the conditions.
Proof of (i i). Set $\gamma = n$ in (i).

Corollary 2. *We have the following identity.*

(i) $$2(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{\Gamma(2k+\gamma)}{\Gamma(\gamma)\Gamma(2k+1)}T^k = \frac{1}{(z-b-\sqrt{c})^{\gamma}}+\frac{1}{(z-b+\sqrt{c})^{\gamma}} \tag{12}$$

$$(|\Gamma(2k+\gamma)| < \infty),$$

and

(i I) $$2(z-b)^{-n}\sum_{k=0}^{\infty}\frac{[2k+1]_{n-1}}{\Gamma(n)}T^k = \frac{1}{(z-b-\sqrt{c})^{n}}+\frac{1}{(z-b+\sqrt{c})^{n}} \tag{13}$$

$$(n \in \mathbf{Z}^+).$$

Proof of (i). Set $m = 1$ in Theorem 2. (i), we have then

$$2\sum_{k=0}^{\infty}\frac{\Gamma(2k+\gamma)}{\Gamma(\gamma)\Gamma(2k+1)}T^k = \sum_{k=0}^{\infty}\frac{\Gamma(k+\gamma)}{\Gamma(\gamma)\Gamma(k+1)}\{(T^{1/2})^k+(-T^{1/2})^k\}. \tag{14}$$

Now we have

$$\sum_{k=0}^{\infty}\frac{\Gamma(k+\gamma)}{\Gamma(\gamma)\Gamma(k+1)}(T^{1/2})^k = \sum_{k=0}^{\infty}\frac{[\gamma]_k}{k!}(T^{1/2})^k \tag{15}$$

$$= \frac{(z-b)^{\gamma}}{(z-b-\sqrt{c})^{\gamma}} . \tag{16}$$

Therefore, we obtain (12) from (14) and (16).
Proof of (i i). Set $\gamma = n$ in (i).

Note. Other proof of (i). We have

$$(\log((z-b)^2-c))_{\gamma} = (\log((z-b)-\sqrt{c}))_{\gamma} + (\log((z-b)+\sqrt{c}))_{\gamma} , \tag{17}$$

from (9), setting $m = 1$.

Next we have

$$(\log((z-b)-\sqrt{c}))_{\gamma} = -e^{-i\pi\gamma}\Gamma(\gamma)(z-b-\sqrt{c})^{-\gamma}, \quad (|\Gamma(\gamma)| < \infty) \tag{18}$$

by Lemma (i i).

Therefore, we obtain (12) from (17) , applying (3) and (18).

Theorem 3. *We have*

(i) $$(\log(z^2+2az+d))_\gamma = -e^{-i\pi\gamma}2(z+a)^{-\gamma}\Gamma(\gamma)$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(2k+\gamma)}{\Gamma(\gamma)\Gamma(2k+1)}\left(\frac{a^2-d}{(z+a)^2}\right)^k \tag{19}$$

$$(|\Gamma(\gamma)|,\ |\Gamma(2k+\gamma)|<\infty),$$

and

(ii) $$(\log(z^2+2az+d))_n = (-1)^{n+1}2(z+a)^{-n}\Gamma(n)$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(2k+n)}{\Gamma(n)\Gamma(2k+1)}\left(\frac{a^2-d}{(z+a)^2}\right)^k \quad (n\in Z^+), \tag{20}$$

where

$$z^2+2az+d\neq 0,1, \quad \textit{and} \quad |(a^2-d)/(z+a)^2|<1 .$$

Proof of (i). We have

$$z^2+2az+d=(z+a)^2-c, \qquad (c=a^2-d). \tag{21}$$

hence

$$\log(z^2+2az+d)=\log((z+a)^2-c) . \tag{22}$$

Operate N^γ to the both sides of (22), we have then

$$(\log(z^2+2az+d))_\gamma=(\log((z+a)^2-c))_\gamma , \tag{23}$$

therefore, we obtain (19) clearly, setting $b=-a$ and $c=a^2-d$ in Corollary 1. (i), under the conditions stated before.

Proof of (ii). Set $\gamma=n$ in (19).

§ 3. Semi Derivatives and Integrals

Corollary 3. *We have*

(i) $$(\log((z-b)^m-c))_{1/2} = im(z-b)^{-1/2}\sqrt{\pi}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(mk+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(mk+1)}\left(\frac{c}{(z-b)^m}\right)^k \tag{1}$$

(semi derivatives),

and

(ii) $$(\log((z-b)^m - c))_{-1/2} = i\,m(z-b)^{1/2}2\sqrt{\pi}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(mk-\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(mk+1)}\left(\frac{c}{(z-b)^m}\right)^k \qquad (2)$$

(semi integrals),

where

$$m \in \mathbf{Z}_0^+ , \quad (z-b)^m - c \neq 0,1, \quad \textit{and} \quad |c/(z-b)^m| < 1 .$$

Proof. Set $\gamma = 1/2$ and $-1/2$ in Theorem 1. (i), we have then (1) and (2) respectively

Corollary 4. *We have*

(i) $$(\log((z-b)^2 - c))_{1/2} = i2\sqrt{\pi}(z-b)^{-1/2} \sum_{k=0}^{\infty} \frac{\Gamma(2k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2k+1)}\left(\frac{c}{(z-b)^2}\right)^k \qquad (3)$$

(semi derivatives),

and

(ii) $$(\log((z-b)^2 - c))_{-1/2} = i4\sqrt{\pi}(z-b)^{1/2} \sum_{k=0}^{\infty} \frac{\Gamma(2k-\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(2k+1)}\left(\frac{c}{(z-b)^2}\right)^k \qquad (4)$$

(semi integrals),

where

$$(z-b)^2 - c \neq 0,1, \quad \textit{and} \quad |c/(z-b)^2| < 1 .$$

Proof. Set $m = 2$ in Corollary 3.

Corollary 5. *We have*

(i) $$(\log((z-b)^2 - c))_{1/2} = i\sqrt{\pi}\left(\frac{1}{\sqrt{z-b-\sqrt{c}}} + \frac{1}{\sqrt{z-b+\sqrt{c}}}\right) \qquad (5)$$

(semi derivatives)

and

(i) $$(\log((z-b)^2 - c))_{-1/2} = i2\sqrt{\pi}\left(\sqrt{z-b-\sqrt{c}} + \sqrt{z-b+\sqrt{c}}\right) \qquad (6)$$

(semi integrals)

where

$$(z-b)^2 - c \neq 0,1.$$

Proof. Set $\gamma = 1/2$ and $-1/2$ in Corollary 2, we have then (5) and (6) respectively, under the conditions.

§ 4. Some Special Cases

[I] When $n=1$, we have

$$(\log((z-b)^2-c))_1 = 2(z-b)^{-1}\sum_{k=9}^{\infty}\left(\frac{c}{(z-b)^2}\right)^k \tag{1}$$

$$= 2(z-b)^{-1}\sum_{k=9}^{\infty}\frac{[1]_k}{k!}\left(\frac{c}{(z-b)^2}\right)^k = 2(z-b)^{-1}\left(1-\frac{c}{(z-b)^2}\right)^{-1} \tag{2}$$

$$= 2((z-b)^2-c)^{-1}(z-b) \quad , \tag{3}$$

from Corollary 1 (i i).

[I I] When $n=2$, we have

$$(\log((z-b)^2-c))_2 = -2(z-b)^{-2}\sum_{k=0}^{\infty}\frac{\Gamma(2k+2)}{\Gamma(2k+1)}\left(\frac{c}{(z-b)^2}\right)^k \tag{4}$$

$$= -2(z-b)^{-2}\sum_{k=0}^{\infty}\frac{[1]_k(2k+1)}{k!}T^k \qquad \left(T=\frac{c}{(z-b)^2}\right) \tag{5}$$

$$= -2(z-b)^{-2}\left\{\sum_{k=0}^{\infty}\frac{[1]_k 2k}{k!}T^k + \sum_{k=0}^{\infty}\frac{[1]_k}{k!}T^k\right\} \tag{6}$$

$$= 2(z-b)^{-2}\{2T(1-T)^{-2}+(1-T)^{-1}\} \tag{7}$$

$$= -2((z-b)^2+c)\,((z-b)^2-c)^{-2} \tag{8}$$

from Corollary 1 (i i).

This result coincides with the one obtained by the classical calculus ;

$$\frac{d^2}{dz^2}\log((z-b)^2-c) \ .$$

Note. $$\sum_{k=0}^{\infty}\frac{[1]_k 2k}{k!}T^k = 2\sum_{k=1}^{\infty}\frac{[1]_k}{(k-1)!}T^k = 2T\sum_{k=0}^{\infty}\frac{[1]_{k+1}}{k!}T^k$$

$$= 2T\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k = 2T(1-T)^{-2}.$$

References

[1] K. Nishimoto ; Fractional Calculus, Vol. 1 (1984), Vol. 2 (1987), Vol. 3 (1989), Vol. 4 (1991), Vol. 5, (1996), Descartes Press, Koriyama, Japan.

[2] K. Nishimoto ; An Essence of Nishimoto's Fractional Calculus (Calculus of the 21st Century); Integrals and Differentiations of Arbitrary Order (1991), Descartes Press, Koriyama, Japan.

[3] K. Nishimoto ; On Nishimoto's fractional calculus operator N^{ν} (On an action group), J. Frac. Calc. Vol. 4, Nov. (1993), 1 - 11.

[4] K. Nishimoto ; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol. 6, Nov. (1994), 1 - 14.

[5] K. Nishimoto ; Ring and Field Produced from The Set of N- Fractional Calculus Operator, J. Frac Calc. Vol. 24, Nov. (2003),29 - 36.

[6] K. Nishimoto ; On the fractional calculus $(a-z)^{\beta}$ and $\log(a-z)$, J. Frac. Calc. Vol.3, May (1993), 19 - 27.

[7] K. Nishimoto and S.- T. Tu ; Fractional calculus of Psi functions (Generalized Polygamma unctions), J. Frac. Calc. Vol.5 May (1994), 27 - 34.

[8] S.- T. Tu and K. Nishimoto ; On the fractional calculus of functions $(cz-a)^{\beta}$ $\log(cz-a)$, J. Frac.Calc.Vol.5, May (1994), 35 - 43.

[9] K. Nishimoto ; N- Fractional Calculus of the Power and Logarithmic Functions and Some Identities, J. Frac. Calc. Vol.21, May (2002), 1 - 6.

[10] K. Nishimoto ; Some Theorems for N- Fractional Calculus of Logerithmic Functions I, J. Frac Calc.Vol.21, May (2002), 7 - 12.

[11] K. Nishimoto ; N- Fractional Calculus of Products of Some Power Functions, J. Frac.Calc. Vol.27, May (2005), 83 - 88.

[12] K. Nishimoto ; N- Fractional Calculus of Some Composite Functions, J. Frac. Calc. Vol. 29, May (2006), 35 - 44.

[13] K. Nishimoto ; N- Fractional Calculus of Some Composite Algebraic Functions,J. Frac.Calc. Vol. 31, May (2006), 11 - 23.

[14] K. Nishimoto ; N- Fractional Calculus of Some Elementary Functions and Their Semi Differintegrations, J. Frac. Caic. Vol. 31, May (2007), 1 - 10.

[15] K. Nishimoto and T. Miyakoda ; N- Fractional Calculus and n-th Derivatives of Some Algebraic Functions, J. Frac. Calc. Vol. 31, May (2007), 53 - 62.

[16] T. Miyakoda ; N- Fractional Calculus of Certain Algebraic Functions, J. Frac. Calc.Vol. 31, May (2007), 63 - 76.

[17] K. B. Oldham and J. Spanier ; The Fractional Calculus, Academic Press (1974).

[18] S. Moriguchi, K.Udagawa and S. Hitotsumatsu ; Mathematical Formulae, Vol.2, Iwanami Zensho, (1957), Iwanami, Japan.

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